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Fewnomials and Pfaff Manifolds

The ideology of fewnomials implies that real varieties, defined by "simple" (not too complicated) sets of equations, must have a simple topology. Of course, this is not always true. The fewnomial ideology, however, is helpful in finding a number of rigorous results.

The classical Bézout theorem states that the number of complex solutions of a set of k polynomial equations in k unknowns can be estimated in terms of their degrees (it equals the product of the degrees). This report is concerned with the real and the transcendental analogues of this theorem: for a wide class of real transcendental equations (including all real algebraic ones) the number of solutions of a set of k such equations in k real unknowns if finite and can be explicitly estimated in terms of the "complexity" of the equations. A more general result involves a construction of a class of transcendental real varieties resembling algebraic varieties.

These results provide new information about polynomial equations (see Sections 1 and 11) and level sets of elementary functions (see Sections 2 and 10).

1. Real fewnomials

The topology of geometric objects determined by algebraic equations (real algebraic curves, surfaces, singularities, etc.) gets more and more complex as the degree of the equation increase. As recently found complexity of the topology depends only on the number of monomials contained in the equations rather than on their degrees: the following Theorems 1 and 2 assess the complexity of the topology of geometrical objects in terms of the complexity of equations determining the object.

We begin with the following well-known

DESCARTES RULE. *The number of positive roots of a polynomial in a single real variable does not exceed the number of sign alternations in the sequence of its coefficients* (null coefficients are deleted from the sequence).

COROLLARY (the Descartes estimate). *The number of positive roots of a polynomial is less than the number of its terms.*

A. G. Kushnirenko proposed to call polynomials with a small number of terms *fewnomials*. The Descartes estimate shows that independently of the degree of a fewnomial (which may be as large as we wish) the number of its positive roots is small.

The following Theorems 1 and 2 (see [9]) generalize the Descartes estimate to the case of systems of polynomial equations in multi-dimensional real space.

Denote by q the number of monomials appearing with nonzero coefficients in at least one of the polynomials of the system.

THEOREM 1. *The number of non-degenerate solutions of a system of k polynomial equations in k positive real unknowns is less than* $2^{q(q-1)/2}(k+1)^q$.

THEOREM 2. *The sum of Betti numbers of a non-singular algebraic manifold defined in R^k by a non-degenerate system of polynomial equations is not greater than an explicitly expressed function of k and q. The number of connected components of a singular algebraic variety can also be estimated from above in terms of k and q.*

The known estimates of the sum of Betti numbers and of the number of connected components in Theorem 2, as well as the estimate of the number of roots in Theorem 1 contain an unpleasant factor of order $2^{q^2/2}$. Apparently, these estimates are far from being exact.

The arguments proving Theorems 1 and 2 are not only useful in algebra. Let us state a result related to the theory of elementary functions.

2. Level surfaces of elementary functions

We begin with definitions. Here is a list of *principal elementary functions*: the exponent, the logarithm, trigonometric functions (sin, cos, tan, cot) and their inverse functions. The function defined in a domain in R^n which can be represented as a composition of a finite number of algebraic functions and principal elementary functions is called *elementary*. An *elementary manifold* is the transversal intersection of non-singular level surfaces of several elementary functions. A map of degree $\leqslant m$ of an elementary manifold in R^n is the restriction to the manifold of such a map of R^n into R^k that all its components are polynomials of degree $\leqslant m$.

Choose a compact subset K in some k-dimensional elementary manifold.

THEOREM. *In any regular value in R^k of a map of degree m of a k-dimensional elementary manifold, the number of points in the inverse image contained in K is less than Cm^r. In this estimate the constant r depends only on the elementary manifold, while the constant C depends on the choice of the set K as well.*

This theorem may be applied to complex level surfaces of elementary functions in several complex variables and to the intersections of such surfaces. The theorem has various generalizations. For example, it remains valid if instead of level surfaces of elementary functions we consider level surfaces of functions which can be represented in quadratures.

COROLLARY. *The number of isolated intersection points of a compact arc of the graph of an elementary function in one variable with a plane algebraic curve of degree m is no greater than Cm^r. If for a compact arc of the graph of a certain function f there exists a sequence of algebraic curves of degrees m_n, whose number of intersection points with the arc increases faster than any power of the numbers m_n, then the function f is necessarily nonelementary.*

A similar situation is known in the theory of numbers: according to the Liouville theorem algebraic numbers have only "slow" approximations by rational numbers, so that if for some number there exists a "rapid" approximation, then this number is necessarily transcendental-

Note that elementary functions have special properties also as multi valued functions of a complex variable [5].

3. Pfaff curves

There exists a wide class of real analytic manifolds whose properties are similar to those of algebraic manifolds. It is precisely this fact that the theorems of the previous section are based on.

Now we pass to the simplest variant of this theory. Consider a dynamic system on a plane given by a polynomial vector field. The trajectories of such a system may differ drastically from algebraic curves. There is no analogue of the Bézout theorem for such lines. Thus, for example, the trajectory of the system winding around a cycle, has a countable number of points in common with any straight line intersecting the cycle. The following constraint on the topology of the trajectory affects its properties and makes them similar to those of an algebraic curve.

DEFINITION. An oriented, smooth (possibly disconnected) curve in a plane is said to be a *separating solution of a dynamic system* if (a) the curve consists of one or several trajectories of the system (with natural orientation of trajectories); (b) the curve does not pass through the singular points of the system; (c) the curve is the boundary of a certain plane domain equipped with a natural boundary orientation.

Examples. (1) The cycle of a dynamic system is always its separating solution: It is oriented either as the boundary of the interior domain with respect to the cycle, or as the outer domain boundary. (2) A non-compact trajectory tending to infinity for $t \to +\infty$ and for $t \to -\infty$ is a separating solution. (3) A non-critical level line of a function $H(x_1, x_2) = c$, oriented as the boundary of the domain $H < c$, is a separating solution of the Hamiltonian system $\dot{x}_1 = -\partial H/\partial x_2$, $\dot{x}_2 = \partial H/\partial x_1$.

DEFINITION. A curve on a plane is called a *Pfaff curve of degree n* if there is an orientation of the curve for which the curve is a separating solution of a dynamic system, given by a vector field whose components are polynomials of degree n.

Smooth algebraic curves of degree $n+1$ are the Pfaff curves of degree n (see Example (3)). Thus the Pfaff curves can be viewed as generalizations of plane algebraic curves.

THEOREM (The analogue of the Bézout theorem for Pfaff curves). (1) *Restrictions of a polynomial of degree m to a Pfaff curve of degree n have at most $m(n+m)$ isolated roots.* (2) *Two Pfaff curves of degrees n and m have at most $(n+m)(2n+m)+n+1$ isolated intersection points.*

Let us consider the direct corollaries of this theorem.

COROLLARIES. (1) *All cycles of a dynamic system with polynomial field of degree 2 are convex.* (2) *Restrictions of a polynomial of degree m to a Pfaff curve of degree n have at most $(n+m-1)(2n+m-1)$ critical values on this curve.* (3) *A Pfaff curve of degree n has at most $n+1$ non-compact components and at most $(3n-1)(4n-1)$ inflection points.*

Of course, these estimates, both in the theorem and the corollaries are not the best possible (this is indicated, in particular, by the asymmetry with respect to n and m in the second statement of the theorem). However, these estimates are not so bad. Thus, for any m and any $n > 0$ it is easy to construct examples where the number of roots in the first statement of the theorem is not smaller than one third of the respective estimate. The estimate of the number of non-compact components is sharp and the estimates of the number of compact components and of the number of inflection points have the same order of growth for $n \to \infty$ as the sharp estimates for algebraic curves of degree $n+1$.

We now pass to the multidimensional case.

4. Separating solutions and Rolle's theorem

Let M be a smooth manifold (possibly disconnected, non-oriented and infinite-dimensional) and let α be a 1-form on it. Of great significance for the sequel is the following generalization of the separating solution of dynamic system on a plane.

DEFINITION. A submanifold of codimension one in M is said to be a *separating solution of the Pfaff equation* $\alpha = 0$ if (a) the restriction of the form α to the submanifold is identically zero; (b) the submanifold does not pass through the singular points of the equation (i.e., at each point of the submanifold the form α does not vanish on the tangent space); (c) the submanifold is a boundary of a domain in M and its coorientation defined by the form coincides with the coorientation of the domain boundary (i.e., on the vectors, applied at the submanifold points and outgoing from the domain, the form α is positive).

Example. The surface $H = c$ of a non-singular level of the function H is a separating solution of the equation $dH = 0$ (it bounds the domain $H < c$).

A *Pfaff hypersurface in* R^n is a separating solution of the equation $\alpha = 0$ where α is a 1-form in R^n with polynomial coefficients. An algebraic hypersurface is a Pfaff hypersurface (see the example). The Pfaff hypersurface resembles an algebraic one in many ways. Suppose β is the restriction of the 1-form with polynomials coefficients to the Pfaff hypersurface. A separating solution of the equation $\beta = 0$ on the Pfaff hypersurface also possesses properties similar to those of an algebraic manifold. This process may be continued. We obtain a wide class of manifolds resembling algebraic ones. The formal definition of this class is given in Sections 5, 7.

Here we will dwell on a certain property of separating solutions. For such solutions we have the following multidimensional variant of Rolle's theorem.

PROPOSITION. *Between two intersection points of a connected smooth curve with a separating solution of a Pfaff equation there is a point of contact, i.e., a point at which the tangent vector to the curve lies in hyperplane* $\alpha = 0$.

The proof is especially easy in the case where the curve intersects the separating solution transversally. In this case, at the neighbouring points of intersection, the values of the form α on the tangent vectors orienting the curve have different signs. Therefore, the form α vanishes at a certain intermediate point.

To demonstrate the significance of Rolle's theorem, we consider a simple transcendental generalization of the Descartes estimate.

PROPOSITION (Laguerre). *The number of real roots of a linear combination of exponents* $\sum_{i=1}^{q} \lambda_i \exp(a_i t)$ *is less than the number of exponents* q.

The Descartes estimate of the number of positive roots of a polynomial follows from the Laguerre proposition by substitution $x = \exp t$.

The proposition is proved by induction. Let us divide the linear combination by one of its exponents and differentiate the quotient. The derivative contains fewer exponents. According to Rolle's theorem, the number of zeroes of the function does not exceed the number of zeroes of the derivative plus 1.

Pfaff manifold theory is something of a multidimensional generalization of this simple argument (unidimensional generalization can be found in [4]).

5. Simple Pfaff manifolds

Let X be a real analytic manifold and A — a certain finitely generated ring of analytical functions on it.

DEFINITION. The following set of objects is called a *simple realization of the pair* (X, A): (a) an embedding $\pi\colon X \to R^n$ such that ring A coincides with the image of the polynomial ring under π^*; (b) a chain of embedded submanifolds in R^n, $X_0 \supset X_1 \supset \ldots \supset X_q$ in which every manifold is a hypersurface in the preceding one, the first manifold X_0 coincides with R^n and the last manifold X_q contains the image of X under the embedding π as one or several of its connected components; (c) a chain of polynomial 1-forms $\alpha_1, \ldots, \alpha_q$ in R^n such that each manifold X_i in the chain is a separating solution of the Pfaff equation $\alpha_i = 0$ on the preceding manifold X_{i-1}.

DEFINITION. A pair (X, A) is called a simple *Pfaff A-manifold* (briefly, A-manifold) if a simple realization for it exists.

DEFINITION. The *complexity of a simple realization of the pair* (X, A) is the set of degrees of all the polynomials which are the coefficients of all 1-forms α_i appearing in its simple realization. By the *degree of a function from ring A* at a simple realization we mean the minimal degree of the polynomial sent to this function by π^*.

In the following two examples, the pair (X, A) is defined simultaneously with its simple realization. The manifold X is defined as a transversal intersection of q non-singular hypersurfaces $f_i = 0$ in R^n, the ring A as the restriction to X of the polynomial ring and the i-th manifold in the chain as the intersection of the first i hypersurfaces (i.e. X_i is determined by the system $f_1 = \ldots = f_i = 0$).

Examples. (1) Consider algebraic hypersurfaces $f_i = 0$ (the functions f_i are polynomials). The chain of forms is $\alpha_i = df_i$. The complexity of realization is determined by the degrees of equations $f_i = 0$ defining the manifold X. (2) Take hypersurfaces defined by equations $f_i = y_i - \exp\langle a_i, t\rangle$ (here $R^n = R^k \times R^q$, $t \in R^k$, $a_i \in R^{k*}$ and y_i is the i-th component of the vector $y \in R^q$). For the chain of forms, we take the forms $\alpha_i = dy_i - y_i\langle a_i, dt\rangle$. The complexity of the realization is determined by the number q of exponents $\exp\langle a_i, t\rangle$ appearing in the definition of the manifold X.

THEOREM (Bézout theorem for simple Pfaff manifolds). *On a simple A-manifold of dimension k the number of non-degenerate solutions of a system $\varphi_1 = \ldots = \varphi_k = 0$, where $\varphi_i \in A$, is finite. The number of solutions is explicitly estimated by the complexity of any realization of the pair (X, A) and in terms of the degrees of the functions φ_i at the realization.*

Let us quote an example of an explicit estimate. Suppose that for a certain realization of the pair (X, A) the codimension of X in R^n equals q, that the degrees of all polynomial coefficients of the forms α_i do not exceed m, and the degrees of functions $\varphi_1, \ldots, \varphi_k$ are equal to $p_1, \ldots, p_k$.

ESTIMATE. *Under the above conditions, the number of solutions in our* *theorem does not exceed*

$$2^{q(q-1)/2}\left[p_1\cdot\ldots\cdot p_k\left(\sum(p_i-1)+mq-1\right)^q\right].$$

Indeed, our proof results in a more accurate but more awkward estimate (the difference is especially significant in the case where the coefficients of different forms a_i have different degrees or have higher degrees but smaller-volume Newton polyhedra). However, there is an unpleasant factor of the order $2^{q^2/2}$ inherent to our technique (improvements relate to the second factor which grows slower).

Let us consider a set of k equations $P_1 = \ldots = P_k = 0$ in k unknowns $(t_1, \ldots, t_k) = t$ in which P_j is a degree p_j polynomial in $k+q$ variables (t, y), where $y = (y_1, \ldots, y_q)$ and $y_i = \exp\langle a_i, t\rangle$.

THEOREM. *The number of non-degenerate real solutions of the system considered is finite and does not exceed*

$$2^{q(q-1)/2}p_1\cdot\ldots\cdot p_k\left(\sum p_i+1\right)^q.$$

To prove it, it suffices to apply the previous theorem to the manifold X from Example 2. Theorem 1 is a corollary of the formulated theorem (derived from it by the substitution of coordinates $x_i = \exp t_i$).

Let us now proceed to general Pfaff manifolds obtained by glueing simple manifolds.

6. Finiteness theorems

In the sequel we shall give definitions of a class of Pfaff real analytic manifolds, functions, forms and maps. A notion of a realization is introduced for each of these objects (one object has many different realizations). To each realization we assign a set of integers called its complexity.

THEOREM (analogue of the Bézout theorem). *The number of points in a Pfaff zero-dimensional manifold is finite and is explicitly estimated in terms of the complexity of any of its realizations.*

A Pfaff manifold is called *affine* if it holds at least one Morse–Pfaff function defining its proper map into the straight line R^1. The choice of such a function together with its realization is called the *realization of the affine manifold.*

FINITENESS THEOREM. *A Pfaff affine variety has the homotopy type of a finite cellular complex. The number of cells is explicitly bounded from above in terms of the complexity of any of its realizations.*

COROLLARY. *The sum of Betti numbers of an affine Pfaff manifold is finite and explicitly bounded in terms of the complexity of any of its realizations.*

Let us proceed to defining the category of Pfaff manifolds.

7. Pfaff manifolds, functions, forms, and maps

We call a ring of analytic functions on an analytic manifold a *base*, if: (a) the ring is finitely generated; (b) for any two different points of the manifold there is a function from the ring having different values at these points; (c) differentials of the ring function generate cotangent spaces in each point of the manifold.

A *Pfaff cover* of a manifold X with the base ring A is, by definition, a representation of X as a finite sum of open sets $X = \bigcup U_i$ together with the rings A_i of analytic functions in the domains U_i such that (a) the ring A_i contains all the functions that are restrictions of functions from the ring A to the domain U_i, (b) all pairs (U_i, A_i) are simple A_i-manifolds.

DEFINITION. A pair (X, A) is called a *Pfaff A-manifold* if it has a Pfaff cover. A function φ on the manifold X is called a *Pfaff A-function* if there is a Pfaff cover for which the restrictions φ_i of φ to the domains U_i lie within the rings A_i.

PROPOSITION. *Let B be any base ring consisting of A-functions on an A-manifold X. Then X is a B-manifold. Moreover, classes of A-functions and B-functions coincide.*

DEFINITION. A manifold with a function ring K is called a *Pfaff manifold* and functions of the ring K are called *Pfaff functions* if for a certain (and hence for any) base ring of functions $A \subset K$ the manifold is an A-manifold and the ring K coincides with the ring of A-functions.

The differential forms lying in the exterior algebra generated by the Pfaff functions and their differentials are referred to as *Pfaff forms*.

A mapping $\varphi\colon X \to Y$ of Pfaff manifolds with rings K_X and K_Y is called a *Pfaff map* if $\varphi^* K_Y \subseteq K_X$.

PROPOSITION. *Suppose that the Pfaff functions $\{f_i\}$ generate a base ring on the manifold Y. The map $\varphi\colon X \to Y$ is a Pfaff map if and only if the functions $\{\varphi^* f_i\}$ are Pfaff functions on X.*

8. Realizations and their complexity

By a *realization of a Pfaff manifold* with a ring K we mean a choice of a base ring $A \subseteq K$, a Pfaff cover $\{U_i, A_i\}$ and simple realizations of pairs U_i, A_i. The set of complexities of these simple realizations is called the *complexity of the realization.*

A *realization of the function* $\varphi \in K$ is such a realization of the manifold that the restriction φ_i of the function φ to each domain U_i is in the ring A_i. The *complexity of a realization of a function* φ is the complexity of the realization of the manifold together with the set of degrees of functions φ_i for the respective simple realizations of pairs U_i, A_i.

By a *realization of a Pfaff map* $\varphi: X \to Y$ we mean a choice of a base of Pfaff functions $\{f_i\}$ on Y together with the choice of realizations of functions $\{f_i\}$ on Y and functions $\{\varphi^* f_i\}$ on X.

A *realization of a Pfaff form* is a choice of its representations in terms of Pfaff functions and their differentials together with the choice of the functions' realizations.

Finally, by the *complexity of a realization* of a map or of a form we mean the set of complexities of the realizations of the functions involved.

9. Operations on Pfaff manifolds

PROPOSITION 1. *On a smooth real algebraic manifold (affine or projective) there exists only one function ring containing a ring of non-singular rational functions on the algebraic manifold and transforming it into a Pfaff manifold. The corresponding algebraic maps are Pfaff maps.*

PROPOSITION 2. *Let an analytic manifold be embedded in a Pfaff manifold. Then, there exists at most one function ring on the analytic manifold transforming it into a Pfaff manifold for which the embeddding is a Pfaff map.*

If the ring under the conditions of Proposition 2 does exist, then the manifold together with that ring is called a *Pfaff sub-manifold.*

In each of the following cases 1–4, the domain in a Pfaff manifold is a Pfaff submanifold (subdomain).

1. The domain consisting of one or several connected components of the manifold.

2. The domain defined by the inequality $f \neq 0$, where f is a Pfaff function.

3. The domain defined by the inequality $f > 0$, where f is a Pfaff function.

4. The domain being the complement to the zero set of a certain Pfaff form.

In each of the following cases 5–6 the submanifold is a Pfaff manifold.

5. The submanifold being an inverse image of a regular value under a Pfaff map.

6. The submamifold being a hypersurface and a separating solution of a Pfaff equation $\alpha = 0$ for a certain Pfaff 1-form α.

Here are two more operations on Pfaff manifolds.

7. The product of a finite number of Pfaff manifolds is a Pfaff manifold. To be exact, the product has a single function ring transforming it into a Pfaff manifold such that the projections onto the factors are Pfaff maps.

8. Let a manifold X, equipped with a function ring A, be covered by a finite number of domains, U_i, and let π_i: $U_i \to X$ be the embedding maps. If all the domains U_i are $(\pi_i^* A)$-manifolds, then X is an A-manifold.

For all cases 1–8 realization of all the manifolds constructed can be explicitly obtained from any realizations of the objects determining the construction. Its complexity is explicitly bounded from above in terms of the realization of the objects involved.

Note that affine and projective (!) real algebraic manifolds are affine Pfaff manifolds, and that operations 1–7 leave the manifolds inside the class of affine manifolds.

10. Properties and examples of Pfaff functions and Pfaff manifolds

1. Pfaff functions form a ring.

2. If a Pfaff function f does not vanish anywhere, then f^{-1} is a Pfaff function.

3. If w_1 and w_2 are Pfaff forms of a higher degree and w_2 does not vanish anywhere, then w_1/w_2 is a Pfaff function.

4. Let a domain in R^n be a Pfaff domain. Then the Pfaff functions in this domain form a differential ring (all partial derivatives of Pfaff functions are Pfaff functions).

5. Superpositions of Pfaff maps are Pfaff maps. In particular, the class of Pfaff functions is closed with respect to superposition.

6. If a vector function $y = (y_1, \ldots, y_k)$ satisfies a non-degenerate set of equations $F(x, y(x)) = 0$ where $F = F_1, \ldots, F_k$ are Pfaff functions, then $y_1, \ldots, y_k$ are Pfaff functions.

The most important property of the class of Pfaff functions is that it is closed with respect to the solution of Pfaff equations. Let us formulate this property more precisely.

Let M^{n+1} and M^n be $(n+1)$-dimensional and n-dimensional Pfaff manifolds, let $\pi\colon M^{n+1}\to M^n$ and $y\colon M^{n+1}\to R^1$ be a Pfaff map and a Pfaff function, α be a Pfaff 1-form on M^{n+1}, $\Gamma\subset M^{n+1}$ a separating solution of a Pfaff equation $\alpha=0$ on M^{n+1} and let $\hat{\pi}$ be the restriction of the map π to Γ.

PROPOSITION. *If the projection $\hat{\pi}$ is a bijective bianalytic correspondence between Γ and M^n, then the function $y\circ\hat{\pi}^{-1}\colon M^n\to R^1$ is a Pfaff function on M^n.*

COROLLARY. *Suppose that the function $y(t)$, defined on a finite or infinite interval of a straight line, satisfies the differential equation $y'=F(t, y)$, where F is a Pfaff function in the plane or in its domain. Then y is a Pfaff function.*

It follows from the corollary that functions $\exp t$ and $\arctan t$ on the straight line, $\ln t$ and t^a on the ray $t>0$, $\arcsin t$ and $\arccos t$ on the interval $-1<t<1$ are Pfaff functions. The functions $\sin t$ and $\cos t$ are not Pfaff functions on the straight line as they have an infinite number of zeroes. However, they are Pfaff functions on any finite interval $a<t<b$. On the interval $0<t<\pi/2$ the functions sin and cos satisfy the equation $y'=\sqrt{1-y^2}$. The complexity of the minimal realization of these functions on the interval (a, b) is proportional to the integer part of the number $(b-a)/\pi$.

The collection of Pfaff functions on an algebraic variety is much wider than that of algebraic functions. Here are examples of Pfaff functions: $\exp\varphi$, $\arctan\varphi$, $\ln f$, f^a, $\arccos g$, $\arcsin g$, $\sinh$, $\cosh$ where φ, f, g, h are algebraic functions, and $f>0$, $-1<g<1$ and $a<h<b$. The polynomials in the above functions are again Pfaff functions; exponents (etc.) of these polynomials are again Pfaff functions, etc. (see [10]).

Non-singular level surfaces of a Pfaff function on algebraic manifold provide non-trivial examples of Pfaff manifolds. A more general example is given by the intersections of level surfaces of different functions. The finiteness theorem reveals that the sum of Betti numbers of such manifolds can be estimated by the complexity of realizations of functions determining them.

Let us return to the algebra.

11. Complex fewnomials

Complex roots of an elementary binomial equation $z^N - 1 = 0$ of degree $N \to \infty$ are equidistributed with respect to arguments. The theorem formulated below shows that a similar phenomenon is observed for a fewnomial system of equations in k variables. The real fewnomial theorem (see Section 1) is one of the manifestations of this.

First, recall some definitions. The *support* of a polynomial $\sum C_\alpha z^\alpha$ depending on k complex variables is the set of degrees of monomials it contains, i.e., the finite set of points α in the integer lattice of space R^k for which the coefficients C_α are not equal to zero.

The *Newton polyhedron* of a polynomial is the convex hull of its support.

We shall denote a non-degenerate set of k polynomial equations in k complex unknowns by $P = 0$. Denote by $T^k = (\varphi_1, \ldots, \varphi_k) \bmod 2\pi$ the torus arguments of space C^k (the j-th coordinate z_j of vector $z \in C^k$ is $z_j = |z_j| \exp i\varphi_j$). Let G be a domain in T^k. We are interested in the number $N(P, G)$ of solutions of the set $P = 0$ for which all coordinates are nonzero, and their arguments lie in the domain G. In the case $G = T^k$ this number is determined by Bernstein's theorem (see [3], [6–8], [13]): it equals the mixed volume of the Newton polyhedra of the equations multiplied by $k!$. Let us denote by $S(P, G)$ the number from Bernstein's theorem multiplied by the ratio of the volume of G to the volume of T^k. For a certain number $\Pi(\Delta, \partial G)$ depending only on the domain G and the Newton polyhedra of the equations, the following theorem holds.

THEOREM ([11]). *There exists a function φ of k and q such that for any non-degenerate system $P = 0$ of equations in k unknowns containing q monomials the following relation holds:*

$$|N(P, G) - S(P, G)| < \varphi(k, q)\,\Pi(\Delta, \partial G).$$

Let us present the definition of the number $\Pi(\Delta, \partial G)$. Let Δ be a domain in R^k determined by the set of inequalities $\{|\langle \alpha, \varphi \rangle| < \pi/2\}$ corresponding to a set of integer vectors α lying in the unions of the supports of the equations. The number $\Pi(\Delta, \partial G)$ is the least number of parallel translates of the domain Δ needed to cover the boundary of G. As corollaries we obtain two old theorems: (1) Bernstein's theorem is obtained for $G = T^k$, since in this case $\Pi(\Delta, \partial G) = 0$; (2) Theorem 1 (Section 1) on real fewnomials is obtained when G contracts to the point $0 \in T^k$; in this case $\Pi(\Delta, \partial G) = 1$, $S(P, G) \to 0$.

For sets of equations with large Newton polyhedra, the number $S(P, Q)$ exceeds, in order of magnitude, the number $\Pi(\Delta, \partial G)$ (see [11]). Therefore, the theorem suggests a uniform distribution of roots of a fewnomial equation with respect to arguments.

A few words about proof of the theorem. First, it is shown that the average number $N(Q, G)$ coincides with the number $S(P, G)$. Averaging is performed for all systems $Q = 0$ whose equations have the same supports as the equations of the original set $P = 0$. This part of the proof is considerably clarified by the Atiyah paper on momentum mappings under almost periodic simplicial actions of the torus ([1], [2]). Then it is shown that numbers $N(Q, G)$ for different systems do not differ much. This part of the proof is based on the theorem from Section 5.

12. Some problems

1. The problem of A. G. Kushnirenko: *Find an exact estimate of the number of real roots of a fewnomial system. Give an example of a fewnomial system with the largest possible number of roots.* (The investigation of fewnomials originated with Kushnirenko's problem. The first result is due K. A. Sevastianov: he estimated the number of zeroes of a fewnomial on an algebraic curve. The first multidimensional result is the theorem on real fewnomials in Section 1.)

2. According to the Descartes rule (see Section 1) a polynomial with a large number of terms has few positive roots if the sign in its coefficient sequence rarely changes. *Find a multidimensional analogue of the Descartes rule* (compare [12]).

3. *Is there any analogue of the Seidenberg–Tarski theorem* [14] *for Pfaff manifolds?* Probably there exists such an analogue for a narrow class of varieties including the class of algebraic varieties. (Added in proof: see [21].)

4. *How can one extend the class of Pfaff manifolds and yet retain the finiteness theorems?* (Added in proof: see [15].)

5. Here is a more specific problem. Let ω be a 1-form in the plane whose coefficients are polynomials of the n-th degree. Cosider the integral J of the form ω with respect to a compact component of the level line $H = c$ of a polynomial H of the $(n+1)$-th degree. When the parameter c ranges over intervals on which the integration curve is not restructured, the integral is an analytic function of the parameter. The problem (of V. I. Arnold) is: *to estimate the number of the isolated integral zeroes as a function of the parameter in this interval.* The Arnold problem is a linearization in the neighbourhood of Hamiltonian systems of Hilbert's sixteenth

problem about the number of limiting cycles of polynomial dynamical systems in the plane. (Equation $J(c) = 0$ is the linearization of the cycle birth condition from the level line $H = c$ of the Hamiltonian H under a perturbation of the Hamiltonian system by the vector field $\varepsilon j(\omega)$, where ε is a small number and j is the isomorphism between cotangent and tangent spaces induced by the standard simplicial structure in the plane.)

(Added in proof: This problem was studied in [16], [17], the latter paper considerably advances its solution.)

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Added in proof:

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